Erdős-Feller-Kolmogorov-Petrowsky law of the iterated logarithm for self-normalized martingales: a game-theoretic approach

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Abstract

We prove an Erdős–Feller–Kolmogorov–Petrowsky law of the iterated logarithm for self-normalized martingales. Our proof is given in the framework of the game-theoretic probability of Shafer and Vovk. As many other game-theoretic proofs, our proof is self-contained and explicit.

Keywords and phrases: Bayesian strategy, constant-proportion betting strategy, lower class, upper class, self-normalized processes.

1 Main Result

Let S_n be a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ and let $x_n = S_n - S_{n-1}$ be the martingale difference. On some regularity conditions on the growth of $|x_n|$, various versions of the law of the iterated logarithm (LIL) have been given in literature. In particular the Erdős–Feller–Kolmogorov–Petrowsky law of the iterated logarithm (EFKP-LIL [16, Chapter 5.2]) is an important extension of LIL. Erdős [6] proved EFKP-LIL for symmetric Bernoulli random variables. EFKP-LIL has been generalized by Feller [7] for bounded and independent random variables and [8] (see also Bai [1]) for the i.i.d. case. Further, EFKP-LIL has been generalized for martingales by Strassen [19], Jain, Jogdeo and Stout [10], Philipp and Stout [15], Einmahl and Mason [5] and Berkes, Hörmann and Weber [2]. In particular, Einmahl and Mason [5] proved a martingale analogue of Feller's result in [7], just as Stout [18] obtained a martingale analogue of Kolmogorov's result in [11].

For self-normalized processes, EFKP-LIL was derived by [9, 3] in the i.i.d. case. However EFKP-LIL has not been derived in the martingale case, even though de la Peña, Klass and Lai [4] obtained the usual LIL. The purpose of this paper is to prove EFKP-LIL for self-normalized martingales. For a positive non-decreasing continuous function $\psi(\lambda)$ let

$$I(\psi) := \int_1^\infty \psi(\lambda) e^{-\psi(\lambda)^2/2} \frac{d\lambda}{\lambda}. \qquad (1)$$

We state our main theorem.

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Theorem 1.1. *Let S_n, $n = 1, 2, \dots$, be a martingale with $S_0 = 0$ and $x_n = S_n - S_{n-1}$ be a martingale difference with respect to a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ such that*

$$|x_n| \le c_n \ a.s.$$

for some $\mathcal{F}_{n-1}$-measurable random variable c_n. Let

$$A_n^2 := \sum_{i=1}^{n} x_i^2 \ \ge 0$$

and let ψ be a positive non-decreasing continuous function.

If $I(\psi) < \infty$, then

$$\mathrm{P}\left(S_n < A_n\psi(A_n^2) \ a.a. \mid \lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty\right) = 1. \tag{2}$$

If $I(\psi) = \infty$, then

$$\mathrm{P}\left(S_n \ge A_n\psi(A_n^2) \ i.o. \mid \lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty\right) = 1. \tag{3}$$

This theorem is a self-normalization of the result in Einmahl and Mason [5] and a generalization of the result in de la Peña, Klass and Lai [4]. The order of growth $A_n/(\psi(A_n^2))^3$ for c_n is currently the best known order for EFKP-LIL even in the independent case ([2]). We call (2) the *validity* and (3) the *sharpness* of EFKP-LIL.

In (2) and (3), we are not assuming that the conditioning events happen with probability one. We can state (2) equivalently as

$$\mathrm{P}\left(\lim A_n = \infty, \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty, S_n \ge A_n\psi(A_n^2) \ i.o.\right) = 0. \tag{4}$$

For our proof we adopt the framework of game-theoretic probability by Shafer and Vovk [17]. In a game-theoretic approach, for proving (2), we explicitly construct a non-negative martingale diverging to infinity on the event of (4).

We use the following notation throughout the paper

$$\ln_k n := \underbrace{\ln \ln \dots \ln}_{k \text{times}} n.$$

We also fix a small positive δ for the rest of this paper, e.g., $\delta = 0.01$. For our proof, as is often seen in the upper-lower class theory (cf. Feller [8, Lemma 1]), we can restrict our attention to ψ such that

$$\psi^L(n) \le \psi(n) \le \psi^U(n) \text{ for all sufficiently large } n, \tag{5}$$

where

$$\psi^L(n) := \sqrt{2\ln_2 n + 3\ln_3 n}, \quad \psi^U(n) := \sqrt{2\ln_2 n + 4\ln_3 n}.$$

Here L means the lower class and U means the upper class. It can be verified that $I(\psi^U) < \infty$ and $I(\psi^L) = \infty$.

The rest of this paper is organized as follows. In Section 2 we give a game-theoretic statement corresponding to our main theorem. In Section 3 we give a proof of the validity and in Section 4 we give a proof of the sharpness.

2 Preliminaries on Game-Theoretic Probability

In order to state a game-theoretic version of Theorem 1.1, consider the following simplified predictably unbounded forecasting game (SPUFG, Section 5.1 of [17]) with the initial capital $\alpha > 0$.

Simplified Predictably Unbounded Forecasting Game
Players: Forecaster, Skeptic, Reality
Protocol:
$\mathcal{K}_0 := \alpha$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $c_n \geq 0$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [-c_n, c_n]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

Usually α is taken to be 1, but in Section 4 we use $\alpha \neq 1$ for notational simplicity.

We prove the following theorem, which implies Theorem 1.1 by Chapter 8 of [17].

Theorem 2.1. *Consider SPUFG. Let ψ be a positive non-decreasing continuous function. If $I(\psi) < \infty$, Skeptic can force*

$$A_n^2 \to \infty \textit{ and } \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \ \Rightarrow\ S_n < A_n \psi(A_n^2) \ \textit{a.a.} \tag{6}$$

and if $I(\psi) = \infty$, Skeptic can force

$$A_n^2 \to \infty \textit{ and } \limsup c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \ \Rightarrow\ S_n \geq A_n \psi(A_n^2) \ \textit{i.o.} \tag{7}$$

We use the same line of arguments as in [14] and Chapter 5 of Shafer and Vovk [17]. We employ a Bayesian mixture of constant-proportion betting strategies. Here we give basic properties of constant-proportion betting strategies.

A constant-proportion betting strategy with betting proportion $\gamma > 0$ sets

$$M_n = \gamma \mathcal{K}_{n-1}.$$

However, $\mathcal{K}_n$ becomes negative if $\gamma x_n < -1$. For simplicity we consider applying the strategy ("keep the account open") as long as $\gamma c_n \leq \delta$ and sets $M_n = 0$ once $\gamma c_n > \delta$ happens ("freeze the account"). Define a stopping time

$$\sigma_\gamma := \min\{n \mid \gamma c_n > \delta\}. \tag{8}$$

Note the monotonicity of σ_γ, i.e., $\sigma_{\gamma'} \geq \sigma_\gamma$ if $\gamma' \leq \gamma$. We denote the capital process of the constant-proportion betting strategy with this stopping time by $\mathcal{K}_n^\gamma$. With the initial capital of $\mathcal{K}_0^\gamma = \alpha$, the value of $\mathcal{K}_n^\gamma$ is written as

$$\mathcal{K}_n^\gamma = \alpha \prod_{i=1}^{\min(n, \sigma_\gamma - 1)} (1 + \gamma x_i).$$

By

$$t - \frac{t^2}{2} - t^2 \times |t| \le \ln(1+t) \le t - \frac{t^2}{2} + t^2 \times |t|$$

for $|t| \le \delta$, taking the logarithm of $\prod_{i=1}^{n}(1+\gamma x_i)$, for $n < \sigma_\gamma$, we have

$$\gamma S_n - \frac{\gamma^2 A_n^2}{2} - \gamma^3 A_n^2 \overline{c}_n \le \ln(\mathcal{K}_n^\gamma/\alpha) \le \gamma S_n - \frac{\gamma^2 A_n^2}{2} + \gamma^3 A_n^2 \overline{c}_n$$

and

$$e^{-\gamma^3 A_n^2 \overline{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2} \le \mathcal{K}_n^\gamma/\alpha \le e^{\gamma^3 A_n^2 \overline{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2}, \tag{9}$$

where

$$\overline{c}_n := \max_{1 \le i \le n} c_i.$$

We also set up some notation for expressing the condition in (6) and (7). An infinite sequence of Forecaster's and Reality's announces $\omega = (c_1, x_1, c_2, x_2, \dots)$ is called a *path* and the set of paths $\Omega = \{\omega\}$ is called the sample space. Define a subset $\Omega_{<\infty}$ of Ω as

$$\Omega_{<\infty} := \left\{ \omega \mid A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} < \infty \right\}.$$

For an arbitrary path $\omega \in \Omega_{<\infty}$ we have

$$\exists C(\omega) < \infty, \exists n_1(\omega), \forall n > n_1(\omega),\ c_n < C(\omega) \frac{A_n}{\psi(A_n^2)^3},\ \psi(A_n^2) \ge 1. \tag{10}$$

The last inequality holds by the lower bound in (5).

3 Validity

We prove the validity in (6) of Theorem 2.1. In this section we let $\alpha = 1$. We discretize the integral in (1) as

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} e^{-\psi(k)^2/2} < \infty. \tag{11}$$

Since $xe^{-x^2/2}$ is decreasing for $x \ge 1$, the function $\lambda \mapsto \frac{\psi(\lambda)}{\lambda} e^{-\psi(\lambda)^2/2}$ is decreasing for λ such that $\psi(\lambda) \ge 1$ and convergences of the integral in (1) and the sum in (11) are equivalent.

The convergence of the infinite series in (11) implies the existence of a non-decreasing sequence of positive reals a_k diverging to infinity ($a_k \uparrow \infty$), such that the series multiplied term by term by a_k is still convergent:

$$Z := \sum_{k=1}^{\infty} a_k \frac{\psi(k)}{k} e^{-\psi(k)^2/2} < \infty.$$

This is easily seen by dividing the infinite series into blocks of sums less than or equal to $1/2^k$ and multiplying the k-th block by k (see also [13, Lemma 4.15]).

For $k \ge 1$ let

$$p_k := \frac{1}{Z} a_k \frac{\psi(k)}{k} e^{-\psi(k)^2/2}$$

and consider the capital process of a countable mixture of constant-proportion strategies

$$\mathcal{K}_n := \sum_{k=1}^{\infty} p_k \mathcal{K}_n^{\gamma_k}, \quad \text{where} \quad \gamma_k := \frac{\psi(k)}{\sqrt{k}}. \tag{12}$$

Note that $\mathcal{K}_n$ is never negative. By the upper bound in (5), as $k \to \infty$ we have

$$\gamma_k \le \frac{\psi^U(k)}{\sqrt{k}} = \sqrt{\frac{2 \ln_2 k + 4 \ln_3 k}{k}} \to 0. \tag{13}$$

We show that $\limsup_n \mathcal{K}_n = \infty$ if a path $\omega \in \Omega_{<\infty}$ satisfies $S_n \ge A_n \psi(A_n^2)$ *i.o.* We bound $Z\mathcal{K}_n$ as

$$Z\mathcal{K}_n \ge \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2 \rfloor} p_k \mathcal{K}_n^{\gamma_k}. \tag{14}$$

At this point we check that all accounts on the right-hand side of (14) are open for sufficiently large n and the lower bound in (9) can be applied to each term of (14) for $\omega \in \Omega_{<\infty}$. We have the following two lemmas.

Lemma 3.1. *Let $\omega \in \Omega_{<\infty}$. Let $C = C(\omega)$ in* (10)*. For sufficiently large n*

$$\bar{c}_n = \max_{1 \le i \le n} c_i < (1+\delta) C \frac{A_n}{\psi(A_n^2)^3}. \tag{15}$$

Proof. Note that the first $n_1(\omega)$ c's i.e., $c_1, \dots, c_{n_1(\omega)}$, do not matter since $\lim_{n\to\infty} A_n/\psi(A_n^2)^3 = \infty$. For $l > n_1(\omega)$, by (10) we have

$$c_l \le C \frac{A_l}{\psi(A_l^2)^3} \le C A_l.$$

Hence c_l such that $A_l \le A_n/\psi(A_n^2)^3$ do not matter in $\bar{c}_n$.

For c_l such that $A_l > A_n/\psi(A_n^2)^3$ we have

$$c_l \le C \frac{A_l}{\psi(A_n^2/\psi(A_n^2)^6)^3} \le C \frac{A_n}{\psi(A_n^2/\psi(A_n^2)^6)^3} = C \frac{A_n}{\psi(A_n^2)^3} \frac{\psi(A_n^2)^3}{\psi(A_n^2/\psi(A_n^2)^6)^3}.$$

But by (5), both $\psi(A_n^2)$ and $\psi(A_n^2/\psi(A_n^2)^6)$ are of the order $\sqrt{2 \ln_2 A_n^2}(1+o(1))$ and $\psi(A_n^2)/\psi(A_n^2/\psi(A_n^2)^6) \to 1$ as $n \to \infty$. Hence (15) holds. □

Lemma 3.2. *Let $\omega \in \Omega_{<\infty}$. For sufficiently large n, $\sigma_{\gamma_k} > n$ for all $k = \lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor, \dots, \lfloor A_n^2 \rfloor$.*

Proof. By the monotonicity of ψ, we have $\gamma_k \le \psi(A_n^2)/\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}$ for $k = \lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor, \dots, \lfloor A_n^2 \rfloor$. Then by the monotonicity of σ_γ, it suffices to show

$$\frac{\psi(A_n^2)}{\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}} \bar{c}_n \le \delta$$

for sufficiently large n. By (15), the left-hand side is bounded from above by

$$\frac{\psi(A_n^2)}{\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}} \times (1+\delta) C \frac{A_n}{\psi(A_n^2)^3} = (1+\delta) C \frac{A_n}{\sqrt{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}} \frac{1}{\psi(A_n^2)^2}.$$

But this converges to 0 as $n \to \infty$. □

By Lemma 3.2 and the lower bound in (9), for sufficiently large n, we have

$$\mathcal{K}_n^{\gamma_k} \geq e^{-\gamma_k^3 A_n^2 \overline{c}_n} e^{\gamma_k S_n - \gamma_k^2 A_n^2/2}, \quad k = \lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor, \ldots, \lfloor A_n^2 \rfloor$$

and $Z\mathcal{K}_n$ can be evaluated from below as

$$\begin{aligned} Z\mathcal{K}_n &\geq Z \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2 \rfloor} p_k \exp(\gamma_k S_n - \frac{\gamma_k^2 A_n^2}{2} - \gamma_k^3 A_n^2 \overline{c}_n) \\ &= \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2 \rfloor} a_k \frac{\psi(k)}{k} \exp(-\frac{\psi(k)^2}{2} + \gamma_k S_n - \frac{\gamma_k^2 A_n^2}{2} - \gamma_k^3 A_n^2 \overline{c}_n) \end{aligned}$$

Now we assume that $S_n \geq A_n \psi(A_n^2)$ *i.o.* for the path $\omega \in \Omega_{<\infty}$. Then for sufficiently large n such that $S_n \geq A_n \psi(A_n^2)$, $\psi(A_n^2)/(\psi(A_n^2) - 1) \leq 1 + \delta$ and $A_n / \left(\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor\right)^{1/2} \leq 1 + \delta$, we evaluate the exponent part by (9) as

$$\begin{aligned} -\frac{\psi(k)^2}{2} + \gamma_k S_n - \frac{\gamma_k^2 A_n^2}{2} &\geq -\frac{\psi(k)^2}{2} + A_n \psi(A_n^2) \frac{\psi(k)}{\sqrt{k}} - \frac{\psi(k)^2}{k} \frac{A_n^2}{2} \\ &= \psi(k) \left(-\frac{1}{2} \left(1 + \frac{A_n^2}{k} \right) \psi(k) + \sqrt{\frac{A_n^2}{k}} \psi(A_n^2) \right) \\ &\geq -\frac{\psi(A_n^2)^2}{2} \left(\sqrt{\frac{A_n^2}{k}} - 1 \right)^2 \geq -\frac{\psi(A_n^2)^2}{2} \left(\frac{A_n^2}{k} - 1 \right)^2 \\ &\geq -\frac{1}{2} \left(\frac{\psi(A_n^2)}{\psi(A_n^2) - 1} \right)^2 \geq -\frac{1}{2} - 2\delta \end{aligned}$$

and by Lemma 3.1

$$\begin{aligned} \gamma_k^3 A_n^2 \overline{c}_n &\leq \frac{\psi(A_n^2)^3}{(\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor)^{3/2}} A_n^2 (1 + \delta) C \frac{A_n}{\psi(A_n^2)^3} \\ &\leq (1 + \delta) C \left(\frac{A_n}{(\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor)^{1/2}} \right)^3 \\ &\leq C(1 + \delta)^4. \end{aligned} \tag{16}$$

For sufficiently large n, we have

$$\psi(A_n^2) \leq \psi^U(A_n^2) < \psi^U(2k) = \sqrt{2 \ln_2 2k + 4 \ln_3 2k} < 2\sqrt{2 \ln_2 k + 3 \ln_2 k} = 2\psi^L(k) \leq 2\psi(k).$$

Thus by (16),

$$\begin{aligned} Z\mathcal{K}_n &\geq \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2 \rfloor} a_k \frac{\psi(k)}{k} \exp\left(-\frac{1}{2} - 2\delta - C(1 + \delta)^4 \right) \\ &\geq a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor} \frac{\psi(A_n^2)}{2A_n^2} \sum_{k=\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}^{\lfloor A_n^2 \rfloor} \exp\left(-\frac{1}{2} - 2\delta - C(1 + \delta)^4 \right) \end{aligned}$$

$$\geq a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor} \frac{\psi(A_n^2)}{2A_n^2}\left(\frac{A_n^2}{\psi(A_n^2)} - 1\right)\exp\left(-\frac{1}{2} - 2\delta - C(1+\delta)^4\right)$$
$$= a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor}\left(\frac{1}{2} - \frac{\psi(A_n^2)}{2A_n^2}\right)\exp\left(-\frac{1}{2} - 2\delta - C(1+\delta)^4\right).$$

Since $a_{\lfloor A_n^2 - A_n^2/\psi(A_n^2)\rfloor} \to \infty$ as $n \to \infty$, we have shown

$$\omega \in \Omega_{<\infty},\ S_n \geq A_n\psi(A_n^2)\ i.o.\ \Rightarrow\ \limsup_{n\to\infty} \mathcal{K}_n = \infty.$$

4 Sharpness

We prove the sharpness in (7) of Theorem 2.1. As in Section 4.2 of [17] and in [13], in order to prove the sharpness, it suffices to show the following proposition.

Proposition 4.1. *Consider SPUFG. Let ψ be a positive non-decreasing continuous function. If $I(\psi) = \infty$, then for each $C > 0$, Skeptic can force*

$$A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} \leq C\ \Rightarrow\ S_n \geq A_n\psi(A_n^2)\ \ i.o. \tag{17}$$

Once we prove this proposition, we can take the mixture over $C = 1, 2, \ldots$. Then the sharpness follows, because for each $\omega \in \Omega_{<\infty}$, there exists $C(\omega)$ satisfying (10). We denote

$$\Omega_C := \left\{\omega \in \Omega \mid A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} < (1-\delta)C\right\},$$
$$\Omega_0 := \left\{\omega \in \Omega \mid \lim_{n\to\infty} A_n^2 < \infty\right\},$$
$$\Omega_{=\infty} := \left\{\omega \in \Omega \mid A_n^2 \to \infty, \limsup_n c_n \frac{\psi(A_n^2)^3}{A_n} = \infty\right\}.$$

We divide our proof of Proposition 4.1 into several subsections. For notational simplicity we use the initial capital of $\alpha = 1 - 2/e = (e-2)/e$ in this section. In Sections 4.1 and 4.2 we only consider γ and n with $n < \sigma_\gamma$. As in Lemma 3.2 for the validity, this condition will be satisfied for sufficiently small γ and relevant n.

4.1 Uniform mixture of constant-proportion betting strategies

We consider a continuous uniform mixture of constant-proportion strategies with the betting proportion $u\gamma$, $2/e \leq u \leq 1$. This is a Bayesian strategy, a similar one to which has been considered in [12].

Define

$$\mathcal{L}_n^\gamma := \int_{2/e}^1 \prod_{i=1}^{\min(n,\sigma_\gamma - 1)} (1 + u\gamma x_i)du, \qquad \mathcal{L}_0^\gamma = \alpha = 1 - e/2.$$

At round $n < \sigma_\gamma$ this strategy bets $M_n = \int_{2/e}^1 u\gamma \prod_{i=1}^{n-1}(1 + u\gamma x_i)du$. Then by induction on $n < \sigma_\gamma$ the capital process is indeed written as

$$\mathcal{L}_n^\gamma = \mathcal{L}_{n-1}^\gamma + M_n x_n = \int_{2/e}^1 \prod_{i=1}^{n-1}(1 + u\gamma x_i)du + x_n \int_{2/e}^1 u\gamma \prod_{i=1}^{n-1}(1 + u\gamma x_i)du$$

$$= \int_{2/e}^{1} \prod_{i=1}^{n} (1 + u\gamma x_i) du.$$

Applying (9), we have

$$e^{-\gamma^3 A_n^2 \bar{c}_n} \int_{2/e}^{1} e^{u\gamma S_n - u^2\gamma^2 A_n^2/2} du \le \mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} \int_{2/e}^{1} e^{u\gamma S_n - u^2\gamma^2 A_n^2/2} du,$$

for $n < \sigma_\gamma$. We further bound the integral in the following lemma.

Lemma 4.2. *For $n < \sigma_\gamma$,*

$$\mathcal{L}_n^\gamma \le \begin{cases} e^{\gamma^3 A_n^2 \bar{c}_n} e^{2\gamma(S_n/e - \gamma A_n^2/e^2)} & \text{if} \quad S_n \le 2\gamma A_n^2/e, & (18) \\ e^{\gamma^3 A_n^2 \bar{c}_n} \min\left\{ e^{S_n^2/(2A_n^2)} \dfrac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n/2} \right\} & \text{if} \quad 2\gamma A_n^2/e < S_n < \gamma A_n^2, & (19) \\ e^{\gamma^3 A_n^2 \bar{c}_n} \min\left\{ e^{S_n^2/(2A_n^2)} \dfrac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n - \gamma^2 A_n^2/2} \right\} & \text{if} \quad S_n \ge \gamma A_n^2. & (20) \end{cases}$$

Proof. Completing the square we have

$$-\frac{1}{2}u^2\gamma^2 A_n^2 + u\gamma S_n = -\frac{\gamma^2 A_n^2}{2}\left(u - \frac{S_n}{\gamma A_n^2}\right)^2 + \frac{S_n^2}{2A_n^2}.$$

Hence by the change of variables

$$v = \gamma A_n \left(u - \frac{S_n}{\gamma A_n^2}\right), \qquad du = \frac{dv}{\gamma A_n},$$

we obtain

$$\begin{aligned} \int_{2/e}^{1} e^{u\gamma S_n - u^2\gamma^2 A_n^2/2} du &= e^{S_n^2/(2A_n^2)} \int_{2/e}^{1} \exp\left(-\frac{\gamma^2 A_n^2}{2}\left(u - \frac{S_n}{\gamma A_n^2}\right)^2\right) du \\ &= e^{S_n^2/(2A_n^2)} \frac{1}{\gamma A_n} \int_{2\gamma A_n/e - S_n/A_n}^{\gamma A_n - S_n/A_n} e^{-v^2/2} dv. \end{aligned}$$

Then for all cases we can bound $\mathcal{L}_n^\gamma$ from above as

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n + S_n^2/(2A_n^2)} \frac{\sqrt{2\pi}}{\gamma A_n}. \tag{21}$$

Without change of variables, we can also bound the integral $\int_{2/e}^{1} g(u) du$, $g(u) := e^{u\gamma S_n - u^2\gamma^2 A_n^2/2}$, directly as

$$\int_{2/e}^{1} g(u) du \le \max_{2/e \le u \le 1} g(u).$$

Note that

$$g(2/e) = e^{2\gamma(S_n/e - \gamma A_n/e^2)}, \quad g(1) = e^{\gamma S_n - \gamma^2 A_n^2/2}. \tag{22}$$

We now consider the following three cases.

Case 1 $S_n \le 2\gamma A_n^2/e$. In this case $S_n/(\gamma A_n^2) \le 2/e$ and by the unimodality of $g(u)$ we have $\max_{2/e\le u\le 1} g(u) = g(2/e)$. Hence (18) follows from (22).

Case 2 $2\gamma A_n^2/e < S_n < \gamma A_n^2$. In this case $\max_{2/e\le u\le 1} g(u) = g(S_n/(\gamma A_n^2)) = e^{S_n^2/(2A_n^2)}$ and $\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{S_n^2/(2A_n^2)}$. Furthermore in this case $S_n^2 < \gamma A_n^2 S_n$ implies $S_n^2/(2A_n^2) < \gamma S_n/2$ and we also have

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n/2}. \tag{23}$$

By (21) and (23), we have (19).

Case 3 $S_n \ge \gamma A_n^2$. Then $S_n/(\gamma A_n^2) \ge 1$ and $\max_{2/e\le u\le 1} g(u) = g(1)$. Hence

$$\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \bar{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2}. \tag{24}$$

By (21) and (24), we have (20).

□

4.2 Buying a process and selling a process

Next we consider the following capital process.

$$\mathcal{Q}_n^\gamma := 2\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma e}. \tag{25}$$

This capital process consists of buying two units of $\mathcal{L}_n^\gamma$ and selling one unit of $\mathcal{K}_n^{\gamma e}$. This combination of selling and buying is essential in the game-theoretic proof of LIL in Chapter 5 of [17] and [14]. However, unlike Chapter 5 of [17] and [14], where a combination of *three* capital processes is used, we only combine *two* capital processes.

We want to bound $\mathcal{Q}_n^\gamma$ from above.

Lemma 4.3. *Let*

$$C_1 := 2e^{\gamma^3 A_n^2 \bar{c}_n} \exp\left(\frac{(2e-1)((1+e^3)\gamma^3 A_n^2 \bar{c}_n + \ln 2)}{(e-1)^2}\right). \tag{26}$$

Then for $n < \sigma_{\gamma e}$,

$$\mathcal{Q}_n^\gamma \le \begin{cases} C_1 & \textit{if} \quad S_n \le \gamma A_n^2/e, & (27)\\ 2e^{\gamma^3 A_n^2 \bar{c}_n} \min\left\{ e^{S_n^2/(2A_n^2)} \dfrac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n} \right\} & \textit{if} \quad \gamma A_n^2/e < S_n < e\gamma A_n^2, & (28)\\ C_1 & \textit{if} \quad S_n \ge e\gamma A_n^2. & (29) \end{cases}$$

Remark 4.4. In this lemma, C_1 depends on $\bar{c}_n$, γ and A_n through $\gamma^3 A_n^2 \bar{c}_n$. However from Section 4.5 on, we evaluate $\gamma^3 A_n^2 \bar{c}_n$ from above by a constant. Hence, C_1 can be also taken to be a constant (cf. (50)) not depending on γ and A_n. Also note that the interval for S_n in (28) is larger than the interval in (19).

Proof. We bound $\mathcal{Q}_n^\gamma = 2\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma e}$ from above in the following three cases:

$$\text{(i) } S_n \le \gamma A_n^2/e, \quad \text{(ii) } \gamma A_n^2/e < S_n < e\gamma A_n^2, \quad \text{(iii) } S_n \ge e\gamma A_n^2,$$

Case (i) In this case $S_n/e - \gamma A_n^2/e^2 \le 0$. Hence (27) follows from (18) and $\mathcal{Q}_n^\gamma \le 2\mathcal{L}_n^\gamma$.

Case (ii) We again use $Q_n^\gamma \le 2\mathcal{L}_n^\gamma$. If $\gamma A_n^2/e < S_n \le 2\gamma A_n^2/e$, then

$$\frac{S_n}{e} - \frac{\gamma A_n^2}{e^2} \le \frac{\gamma A_n^2}{e^2} \le \frac{S_n}{e}$$

and $\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \overline{c}_n} e^{2\gamma S_n/e} \le e^{\gamma^3 A_n^2 \overline{c}_n} e^{\gamma S_n}$ from (18). Otherwise (28) follows from (19) and (20).

Case (iii) Since $S_n \ge eA_n^2\gamma > A_n^2\gamma$, by (24) we have $\mathcal{L}_n^\gamma \le e^{\gamma^3 A_n^2 \overline{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2}$ and

$$\begin{aligned}
Q_n^\gamma &\le 2\mathcal{L}_n^\gamma - \mathcal{K}_n^{\gamma e} \le 2e^{\gamma^3 A_n^2 \overline{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2} - e^{-\gamma^3 e^3 A_n^2 \overline{c}_n} e^{\gamma e S_n - \gamma^2 e^2 A_n^2/2} \\
&= 2e^{\gamma^3 A_n^2 \overline{c}_n} e^{\gamma S_n - \gamma^2 A_n^2/2} \left(1 - \frac{1}{2} e^{-(1+e^3)\gamma^3 A_n^2 \overline{c}_n} e^{\gamma(e-1)S_n - (e^2-1)\gamma^2 A_n^2/2}\right).
\end{aligned}$$

Hence if the right-hand side is non-positive we have $Q_n^\gamma \le 0$:

$$\begin{aligned}
&S_n \ge eA_n^2\gamma \ \text{ and } \ -(1+e^3)\gamma^3 A_n^2 \overline{c}_n - \ln 2 + \gamma(e-1)S_n - \frac{1}{2}(e^2-1)\gamma^2 A_n^2 \ge 0 \\
&\qquad \Rightarrow \ Q_n^\gamma \le 0.
\end{aligned} \tag{30}$$

Otherwise, write $B_n := (1+e^3)\gamma^3 A_n^2 \overline{c}_n + \ln 2$ and consider the case

$$\gamma(e-1)S_n - \frac{1}{2}(e^2-1)\gamma^2 A_n^2 \le B_n.$$

Dividing this by $e-1$ and also considering $S_n \ge eA_n^2\gamma$, we have

$$\gamma S_n - \frac{1}{2}(e+1)\gamma^2 A_n^2 \le \frac{B_n}{e-1}, \tag{31}$$

$$-S_n + eA_n^2\gamma \le 0. \tag{32}$$

$\gamma \times (32) + (31)$ gives

$$\frac{1}{2}(e-1)\gamma^2 A_n^2 \le \frac{B_n}{e-1} \quad \text{or} \quad \frac{1}{2}\gamma^2 A_n^2 \le \frac{B_n}{(e-1)^2}.$$

Then by (31)

$$\gamma S_n - \frac{1}{2}\gamma^2 A_n^2 \le \frac{B_n}{e-1} + \frac{e}{2}\gamma^2 A_n^2 \le \frac{B_n}{e-1} + \frac{eB_n}{(e-1)^2} = \frac{(2e-1)B_n}{(e-1)^2}.$$

Hence just using $Q_n^\gamma \le 2\mathcal{L}_n^\gamma$ and (24) in this case, we obtain

$$Q_n^\gamma \le 2e^{\gamma^3 A_n^2 \overline{c}_n} \exp\left(\frac{(2e-1)((1+e^3)\gamma^3 A_n^2 \overline{c}_n + \ln 2)}{(e-1)^2}\right) = C_1. \tag{33}$$

This also covers (30) and we have (33) for the whole case (iii).

□

4.3 Change of time scale and dividing the rounds into cycles

For proving the sharpness we consider the change of time scale from λ to k:

$$\lambda = e^{5k \ln k} = k^{5k}.$$

By taking the derivative of $\ln \lambda = 5k \ln k$, we have $d\lambda/\lambda = 5(\ln k+1)dk$. Since $\ln k$ is dominant in $(\ln k+1)$, the integrability condition is written as

$$\int_1^\infty \psi(\lambda) e^{-\psi(\lambda)^2/2} \frac{d\lambda}{\lambda} = \infty \ \Leftrightarrow \ \int_1^\infty (\ln k) \psi(e^{5k \ln k}) e^{-\psi(e^{5k \ln k})^2/2} dk = \infty.$$

Let $f(x) := \psi(e^{5x \ln x}) e^{-\psi(e^{5x \ln x})^2/2}$. Since $xe^{-x^2/2}$ is decreasing for $x \geq 1$, the function $f(x)$ is decreasing for x such that $\psi(e^{5x \ln x}) \geq 1$. Thus, for sufficiently large k and x such that $k \leq x \leq k+1$, we have

$$\frac{1}{2} \ln(k+1) f(k+1) \leq \ln k f(x+1) \leq \ln x f(x) \leq \ln(k+1) f(x) \leq 2 \ln k f(k).$$

Hence, we have

$$\int_1^\infty (\ln k) \psi(e^{5k \ln k}) e^{-\psi(e^{5k \ln k})^2/2} dk = \infty \ \ \Leftrightarrow \ \ \sum_{k=1}^\infty (\ln k) \psi(e^{5k \ln k}) e^{-\psi(e^{5k \ln k})^2/2} = \infty.$$

Then, it suffices to show (17) if $\sum_{k=1}^\infty (\ln k) \psi(e^{5k \ln k}) e^{-\psi(e^{5k \ln k})^2/2} = \infty$.

As in Chapter 5 of [17] and [14], we divide the time axis into "cycles". However, unlike in Chapter 5 of [17] and [14], our cycles are based on stopping times. Let

$$n_k := k^{5k}, \quad k = 1, 2, \ldots, \tag{34}$$

and define a family of stopping times

$$\tau_k := \min\left\{ n \mid A_n^2 \geq n_k \right\}. \tag{35}$$

We define the k-th cycle by $[\tau_k, \tau_{k+1}]$, $k \geq 1$. Note that τ_k is finite for all k if and only if $A_n^2 \to \infty$. Betting strategy for the k-th cycle is based on the following betting proportion:

$$\gamma_k := \frac{\psi(n_{k+1})}{\sqrt{n_{k+1}}} k^2. \tag{36}$$

Note that γ_k in (36) is slightly different from (12).

For the rest of this section, we check the growth of various quantities along the cycles. Let $\omega \in \Omega_C$. For sufficiently large n,

$$|x_n| \leq c_n \leq C \frac{A_n}{\psi(A_n^2)^3}. \tag{37}$$

Furthermore $A_n^2 = A_{n-1}^2 + x_n^2$. This allows us to bound x_n^2 and A_n^2 in terms of A_{n-1}^2. By squaring (37) we have

$$x_n^2 \leq C^2 \frac{A_{n-1}^2}{\psi(A_n^2)^6 - C^2} \tag{38}$$

and

$$A_n^2 = A_{n-1}^2 + x_n^2 \leq A_{n-1}^2 (1 + \frac{C^2}{\psi(A_n^2)^6 - C^2}) = A_{n-1}^2 \frac{\psi(A_n^2)^6}{\psi(A_n^2)^6 - C^2}. \tag{39}$$

Since $\psi(A_n^2)^6/(\psi(A_n^2)^6 - C^2) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{A_n^2}{A_{n-1}^2} = 1.$$

Note that $A_{\tau_k - 1}^2 < n_k \le A_{\tau_k}^2$ by the definition of τ_k. Hence for $\omega \in \Omega_C$ we also have

$$\lim_{k\to\infty} \frac{A_{\tau_k}^2}{n_k} = 1. \tag{40}$$

The limits in the following lemma will be useful for our argument.

Lemma 4.5. *For $\omega \in \Omega_C$*

$$\lim_{k\to\infty} \frac{\psi^U(n_k)}{\psi(n_{k+1})} = 1, \quad \lim_{k\to\infty} \frac{k^5 A_{\tau_k}^2}{n_{k+1}} = e^{-5}, \quad \lim_{k\to\infty} \gamma_k A_{\tau_k} \psi(n_{k+1}) = 0. \tag{41}$$

Proof. All of $\psi^U(n_k)$, $\psi^U(n_{k+1})$, $\psi^L(n_k)$, $\psi^L(n_{k+1})$, $\psi(n_{k+1})$, $\psi(n_{k+1}/k^4)$ are of the order

$$\sqrt{2 \ln \ln e^{5k \ln k}}(1 + o(1)) = \sqrt{2 \ln k}(1 + o(1)) \tag{42}$$

as $k \to \infty$ and the first equality holds by (5). The second equality holds by (40) and

$$\lim_{k\to\infty} \frac{k^5 n_k}{n_{k+1}} = \lim_{k\to\infty} \frac{k^{5(k+1)}}{(k+1)^{5(k+1)}} = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)^{5(k+1)} = e^{-5}.$$

Then $A_{\tau_k}^2/n_{k+1} = (1 + o(1))n_k/n_{k+1} = O(k^{-5})$ and the third equality holds by

$$\gamma_k A_{\tau_k} \psi(n_{k+1}) \le \psi(n_{k+1})^2 k^2 ((1+\delta) n_k/n_{k+1})^{1/2} \to 0 \qquad (k \to \infty).$$

□

4.4 Stopping times for aborting and sequential freezing for each cycle

In (48) of the next section we will introduce another capital process $\mathcal{M}_n^{\gamma_k,k}$, which will be employed in each cycle. Here we introduce some stopping times for aborting the cycle and for sequential freezing of accounts in $\mathcal{M}_n^{\gamma_k,k}$.

We say that we *abort* the k-th cycle, when we freeze all accounts in the k-th cycle and wait for the $(k+1)$-st cycle. There are two cases for aborting the k-th cycle. The first case is when some c_n is too large for $\omega \in \Omega_C$. Define

$$\sigma_{k,C} := \min\left\{ n \ge \tau_k \mid c_n \psi(A_{\tau_k}^2)^3 > (1+\delta) C A_{n-1} \right\}. \tag{43}$$

We will abort the k-th cycle if $\sigma_{k,C} < \tau_{k+1}$. Note that for $\omega \in \Omega_C$, there exists $k_1(\omega)$ such that

$$\sigma_{k,C} = \infty, \quad \text{for } k \ge k_1(\omega). \tag{44}$$

Another case is when S_n is too large. Define

$$\nu_k := \min\{ n \ge \tau_k \mid A_n \psi(A_n^2) < S_n \}. \tag{45}$$

If $\nu_k < \tau_{k+1}$, then Skeptic is happy to abort the k-th cycle, because he wants to force $S_n \geq A_n\psi(A_n^2)$ *i.o.* The above two stopping times will be used in the final construction of a dynamic strategy in Section 4.6.

For each cycle, we define another family of stopping times indexed by $w = 1, \ldots, \lceil \ln k \rceil$, by

$$\tau_{k,w} := \min\left\{ n \mid A_n^2 \geq e^{2(w+2)} \frac{n_{k+1}}{k^4} \right\}. \tag{46}$$

for sequential freezing of accounts of $\mathcal{M}_n^{\gamma_k,k}$ in (48). We have $\tau_k \leq \tau_{k,w}$ for $k \geq 1$ and $w \geq 1$, because

$$\frac{n_{k+1}}{k^4} = \frac{(k+1)^{5(k+1)}}{k^4} > k^{5k} = n_k.$$

Lemma 4.6. *Let $\omega \in \Omega_C$. $\tau_{k,\lceil \ln k \rceil} \leq \tau_{k+1}$ for sufficiently large k.*

Proof. By $A^2_{\tau_{k,w}-1} \leq e^{2(w+2)} n_{k+1}/k^4$ and by (38), for sufficiently large k we have

$$x^2_{\tau_{k,w}} \leq (1+\delta)C^2 \frac{A^2_{\tau_{k,w}-1}}{\psi(A^2_{\tau_k})^6} \leq \frac{(1+\delta)C^2}{\psi(A^2_{\tau_k})^6} \times \frac{e^{2(w+2)} n_{k+1}}{k^4}$$

and

$$A^2_{\tau_{k,w}} \leq A^2_{\tau_{k,w}-1} + x^2_{\tau_{k,w}} \leq (1+\delta)e^{2(w+2)} \frac{n_{k+1}}{k^4}. \tag{47}$$

Then

$$A^2_{\tau_{k,\lceil \ln k \rceil}} \leq (1+\delta)\left(e^{2(\ln k+2)} \frac{n_{k+1}}{k^4}\right) = (1+\delta)e^4 \frac{n_{k+1}}{k^2} \leq n_{k+1} \leq A^2_{\tau_{k+1}}.$$

□

We also compare $\tau_{k,w}$ to $\sigma_{\gamma_k e^{-w+1}}$ defined in (8). This is needed for applying the bounds derived in previous sections to $\mathcal{M}_n^{\gamma_k,k}$ in the next section.

Lemma 4.7. *Let $\omega \in \Omega_C$. $\tau_{k,w} \leq \sigma_{\gamma_k e^{-w+1}}$ for sufficiently large k.*

Proof. By (47) and by Lemma 3.1, for sufficiently large k

$$\gamma_k e^{-w+1} \bar{c}_{\tau_{k,w}} \leq \frac{\psi(n_{k+1})}{\sqrt{n_{k+1}}} k^2 e^{-w+1} \times (1+\delta)^2 C \frac{e^{w+2}\sqrt{n_{k+1}}}{k^2 \psi(A^2_{\tau_k})^3} \leq (1+\delta)^2 C e^3 \frac{\psi(n_{k+1})}{\psi(A^2_{\tau_k})^3} \leq \delta,$$

because $\psi(n_{k+1})/\psi(A^2_{\tau_k})^3 \to 0$ as $k \to \infty$ by (42). □

4.5 Further discrete mixture of processes for each cycle with sequential freezing

We introduce another discrete mixture of capital process for the k-th cycle. Define

$$\mathcal{M}_n^{\gamma_k,k} := \frac{1}{\lceil \ln k \rceil} \sum_{w=1}^{\lceil \ln k \rceil} \mathcal{Q}^{\gamma_k e^{-w}}_{\min(n,\tau_{k,w})} = \frac{1}{\lceil \ln k \rceil} \sum_{w=1}^{\lceil \ln k \rceil} (2\mathcal{L}^{\gamma_k e^{-w}}_{\min(n,\tau_{k,w})} - \mathcal{K}^{\gamma_k e^{-w+1}}_{\min(n,\tau_{k,w})}). \tag{48}$$

Note that the w-th account in the sum of $\mathcal{M}_n^{\gamma_k,k}$is frozen at the stopping time $\tau_{k,w}$. This is needed since the bound for c_n is growing even during the k-th cycle.

In order to bound $\mathcal{M}_n^{\gamma_k,k}$, we first bound C_1 in (26) for each w in the sum of (48) by a constant independent of n. Note that we only need to consider $n \leq \tau_{k,w}$ for the w-th account.

Lemma 4.8. *Let $\omega \in \Omega_C$. $(\gamma_k e^{-w})^3 A_n^2 \overline{c}_n$ and hence C_1 are bounded from above by*

$$(\gamma_k e^{-w})^3 A_n^2 \overline{c}_n \leq (1+\delta)^5 C e^6, \tag{49}$$
$$C_1 \leq 2e^{(1+\delta)^5 C e^6} \exp\left(\frac{(2e-1)((1+\delta)^5 C e^6 (1+e^3) + \ln 2)}{(e-1)^2}\right) =: \overline{C}_1, \tag{50}$$

for sufficiently large k.

Proof. By (42), for sufficiently large k

$$\frac{\psi(n_{k+1})}{\psi(A^2_{\tau_{k,w}})} \leq \frac{\psi(n_{k+1})}{\psi(n_k)} \leq 1+\delta. \tag{51}$$

Thus

$$\begin{aligned}
\gamma_k^3 e^{-3w} A^2_{\min(n,\tau_{k,w})} \overline{c}_{\min(n,\tau_{k,w})} &\leq \gamma_k^3 e^{-3w} \times A^2_{\tau_{k,w}} \times \overline{c}_{\min(n,\tau_{k,w})} \\
&\leq \frac{\psi(n_{k+1})^3}{n_{k+1}^{3/2}} k^6 e^{-3w} \times A^2_{\tau_{k,w}} \times (1+\delta) C \frac{A_{\tau_{k,w}}}{\psi(A^2_{\tau_k})^3} \\
&\leq (1+\delta) C \frac{\psi(n_{k+1})^3}{\psi(A^2_{\tau_k})^3} k^6 e^{-3w} \frac{A^3_{\tau_{k,w}}}{n_{k+1}^{3/2}} \leq (1+\delta)^5 C e^6.
\end{aligned}$$

□

Lemma 4.9. *Let $\omega \in \Omega_C$. For sufficiently large k,*

$$\mathcal{M}_n^{\gamma_k,k} \leq \overline{C}_1 + \frac{2}{\lceil \ln k \rceil} e^{(1+\delta)^5 C e^6} \max_{\gamma \in [\gamma_k/k, \gamma_k]} \left(\min\{ e^{S_n^2/(2n)} \frac{\sqrt{2\pi}}{\gamma A_n}, e^{\gamma S_n} \} \right), \quad n \in [\tau_k, \tau_{k+1}], \tag{52}$$

where $\overline{C}_1$ is given by the right-hand side of (50).

Proof. We have $|\gamma_k e^{-w} \overline{c}_{\min(n,\tau_{k,w})}| \leq |\gamma_k e^{-w+1} \overline{c}_{\min(n,\tau_{k,w})}| \leq \delta$ by Lemma 4.7. Then we can complete the proof of (52) by Lemma 4.3 and Lemma 4.7 because the length of the interval

$$\left\{ w \mid \frac{S_n}{ne} < \gamma e^{-w} < \frac{S_n e}{n} \right\}$$

is equal to 2. □

As in Chapter 5 of Shafer and Vovk [17], we use $\mathcal{M}_n^{\gamma_k,k}$ in the following form.

$$\mathcal{N}_n^{\gamma_k,D} := \alpha + \frac{1}{D} \lceil \ln k \rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2} (\alpha - \mathcal{M}^{\gamma_k,k}_{n-\tau_k}), \quad \alpha = 1 - \frac{2}{e}, \ D = \frac{24\sqrt{2\pi} e^{(1+\delta)^5 e^6 C} + 4\overline{C}_1}{\alpha}. \tag{53}$$

Here we give a specific value of D for definiteness, but from the proof below it will be clear that any sufficiently large D can be used. Since the strategy for $\mathcal{M}^{\gamma_k,k}_{n-\tau_k}$ is applied only to x_n's in the cycle, $\alpha = \mathcal{N}^{\gamma_k,D}_{\tau_k} = \mathcal{M}_0^{\gamma_k}$. Concerning $\mathcal{N}_n^{\gamma_k,D}$ we prove the following two propositions.

Proposition 4.10. *Let $\omega \in \Omega_C$. Suppose that*

$$-A_n \psi^U(A_n^2) \leq S_n \leq A_n \psi(A_n^2), \qquad \forall n \in [\tau_k, \tau_{k+1}]. \tag{54}$$

and $\tau_{k+1} < \sigma_{k,C}$. *Then for sufficiently large* k

$$\mathcal{N}_n^{\gamma_k,D} \geq \frac{\alpha}{2}, \qquad \forall n \in [\tau_k, \tau_{k+1}], \tag{55}$$

and

$$\mathcal{N}_{\tau_{k+1}}^{\gamma_k,D} \geq \alpha\left(1 + \frac{1-\delta}{D}\lceil \ln k\rceil \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2}\right). \tag{56}$$

Proof. In our proof we denote $t = n - \tau_k$, $S_t = S_n - S_{\tau_k}$ and $A_t^2 = A_n^2 - A_{\tau_k}^2$ for $n > \tau_k$. For proving (55), we use (52) for S_t. We bound $\mathcal{M}_t^{\gamma_k,k}$ from above. By the term $\frac{2}{\lceil \ln k\rceil}$ on the right-hand side of (52), it suffices to show

$$\begin{aligned}
&S_t \leq A_{\tau_k}\psi^U(A_{\tau_k}^2) + \sqrt{A_{\tau_k}^2 + A_t^2}\,\psi(A_{\tau_k}^2 + A_t^2) \\
&\quad \Rightarrow \ \psi(n_{k+1}) e^{-\psi(n_{k+1})^2/2} 2e^{(1+\delta)^5 e^6 C} \min\{e^{S_t^2/(2A_t^2)}\frac{\sqrt{2\pi}}{\gamma A_t}, e^{\gamma S_t}\} \leq \frac{D\alpha}{4}, \ \forall \gamma \in [\gamma_k/k, \gamma_k], \ \forall t \in [0, \tau_{k+1} - \tau_k]
\end{aligned}$$

for sufficient large k. Let

$$c_1 = \frac{9}{(1+2\delta)^2} \quad \text{s.t.} \quad \frac{1}{2} - \frac{1}{\sqrt{c_1}} - \delta > 0. \tag{57}$$

We distinguish two cases:

$$\text{(a) } A_t^2 \leq \frac{\psi(n_{k+1})^2}{c_1\gamma^2}, \quad \text{(b) } \frac{\psi(n_{k+1})^2}{c_1\gamma^2} < A_t^2 \leq A_{\tau_{k+1}}^2 - A_{\tau_k}^2.$$

For case (a), $A_{\tau_k}\psi^U(A_{\tau_k}^2) \leq (1+\delta)A_{\tau_k}\psi(n_{k+1})$ by the first equality in Lemma 4.5 for sufficiently large k. Also $\psi(A_{\tau_k}^2 + A_t^2) \leq \psi(n_{k+1})$. Hence in this case

$$\gamma S_t \leq \left((1+\delta)\gamma A_{\tau_k} + \sqrt{\gamma^2 A_{\tau_k}^2 + \psi(n_{k+1})^2/c_1}\right)\psi(n_{k+1}).$$

Then for $\gamma \leq \gamma_k$ by the third equality in Lemma 4.5

$$\gamma S_t \leq \left((1+\delta)\gamma_k A_{\tau_k} + \sqrt{\gamma_k^2 A_{\tau_k}^2 + \psi(n_{k+1})^2/c_1}\right)\psi(n_{k+1}) = \psi(n_{k+1})^2\left(\frac{1}{\sqrt{c_1}} + \delta\right) \tag{58}$$

for sufficiently large k. Since

$$\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} 2e^{(1+\delta)^5 e^6 C} e^{\gamma S_t} \leq \psi(n_{k+1}) \exp\left(-\psi(n_{k+1})^2 (\frac{1}{2} - \frac{1}{\sqrt{c_1}} - \delta)\right) 2e^{(1+\delta)^5 e^6 C} \to 0 \quad (k \to \infty),$$

we have $\mathcal{N}_n^{\gamma_k,D} \geq \alpha/2$ uniformly in $\gamma \in [\gamma_k/k, \gamma_k]$.

For case (b), $\psi(n_{k+1})/\sqrt{c_1} < \gamma A_t$ and $S_t \leq \left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_t^2}\right)\psi(n_{k+1})$. Hence

$$\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \times 2e^{(1+\delta)^5 e^6 C} e^{S_t^2/(2A_t^2)} \frac{\sqrt{2\pi}}{\gamma A_t}$$

$$\leq \psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \times \frac{2e^{(1+\delta)^5 e^6 C}\sqrt{2\pi}\sqrt{c_1}}{\psi(n_{k+1})} \exp\left(\frac{\left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_t^2}\right)^2}{2A_t^2}\psi(n_{k+1})^2\right)$$

$$= 2e^{(1+\delta)^5 e^6 C}\sqrt{2\pi}\sqrt{c_1}\exp\left(\frac{(1+(1+\delta)^2)A_{\tau_k}^2 + 2(1+\delta)A_{\tau_k}\sqrt{A_{\tau_k}^2 + A_t^2}}{2A_t^2}\psi(n_{k+1})^2\right). \tag{59}$$

For $\gamma \leq \gamma_k$,

$$\frac{\psi(n_{k+1})^2}{c_1\gamma^2} < A_t^2 \ \Rightarrow\ \frac{A_{\tau_k}^2}{A_t^2}\psi(n_{k+1})^2 < c_1\gamma^2 A_{\tau_k}^2 \leq c_1\gamma_k^2 A_{\tau_k}^2 = c_1\frac{A_{\tau_k}^2}{n_{k+1}}k^4\psi(n_{k+1})^2 = O(k^{-1}\ln k).$$

Hence $\psi(n_{k+1})^2 A_{\tau_k}^2/A_t^2 \to 0$ as $k \to \infty$. Similarly $\psi(n_{k+1})^2 A_{\tau_k}/A_t \to 0$ as $k \to \infty$, because $\psi(n_{k+1})^2 A_{\tau_k}/A_t = O(k^{-1/2}(\ln k)^{3/2})$. Therefore the right-hand side of (59) is bounded from above by $2e^{(1+\delta)^5 e^6 C}\sqrt{2\pi}\sqrt{c_1}(1+\delta)$ for sufficiently large k and

$$\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \times 2e^{(1+\delta)^5 e^6 C} e^{S_t^2/(2A_t^2)}\frac{\sqrt{2\pi}}{\gamma A_t} \leq \frac{D\alpha}{4},$$

with the choice of D in (53) and c_1 in (57). This proves (55).

Now we prove (56). We focus on the w-th account when $n \geq \tau_{k,w}$. Recall that in this proof we have been denoting $A_t^2 = A_n^2 - A_{\tau_k}^2$. Similarly we denote $A_{\tau_{k,w}}^2$ instead of $A_{\tau_{k,w}}^2 - A_{\tau_k}^2$. Thus

$$e^{2(w+2)}\frac{n_{k+1}}{k^4} - A_{\tau_k}^2 \leq A_{\tau_{k,w}}^2. \tag{60}$$

We will show that $\limsup_{k\to\infty} \mathcal{M}_{\tau_{k+1}-\tau_k}^{\gamma_k,k} \leq 0$, if

$$S_{\tau_{k,w}} \leq A_{\tau_k}\psi(A_{\tau_k}^2) + A_{\tau_{k,w}}\psi(A_{\tau_{k,w}}^2) \leq \psi(n_{k+1})\left\{A_{\tau_k} + A_{\tau_{k,w}}\right\} \leq 2\psi(n_{k+1})A_{\tau_{k,w}}. \tag{61}$$

We evaluate

$$\mathcal{L}_{\tau_{k,w}}^{\gamma_k e^{-w},k} := \int_{2/e}^{1}\exp\left(u\gamma_k e^{-w}S_{\tau_{k,w}} - u^2\gamma_k^2 e^{-2w}A_{\tau_{k,w}}^2/2\right)du$$

from above. Because $u\gamma_k e^{-w}S_{\tau_{k,w}} - u^2\gamma_k^2 e^{-2w}A_{\tau_{k,w}}^2/2$ is maximized at $u = S_{\tau_{k,w}}/(\gamma_k e^{-w}A_{\tau_{k,w}}^2)$ and

$$\frac{S_{\tau_{k,w}}}{\gamma_k e^{-w}A_{\tau_{k,w}}^2} \leq \frac{2\psi(n_{k+1})A_{\tau_{k,w}}}{(\psi(n_{k+1})k^2/\sqrt{n_{k+1}})e^{-w}A_{\tau_{k,w}}^2} \leq \frac{2\sqrt{n_{k+1}}}{k^2 e^{-w}A_{\tau_{k,w}}} \leq \frac{2}{e^2} \leq \frac{2}{e},$$

the integrand in $\mathcal{L}_{\tau_{k,w}}^{\gamma_k e^{-w},k}$ is maximized at $2/e$ and we have

$$\mathcal{L}_{\tau_{k,w}}^{\gamma_k e^{-w},k} \leq \exp\left(\frac{2}{e}\gamma_k e^{-w}S_{\tau_{k,w}} - \frac{2\gamma_k^2 e^{-2w}A_{\tau_{k,w}}^2}{e^2}\right).$$

By (60) and (61), for sufficiently large k,

$$\frac{2}{e}\gamma_k e^{-w}S_{\tau_{k,w}} - \frac{2\gamma_k^2 e^{-2w}A_{\tau_{k,w}}^2}{e^2} \leq \frac{4\gamma_k\psi(n_{k+1})A_{\tau_{k,w}}}{e^{w+1}} - \frac{2\gamma_k^2 A_{\tau_{k,w}}^2}{e^{2(w+1)}}$$

$$
\begin{aligned}
&= \frac{\psi(n_{k+1})^2 k^2 A_{\tau_{k,w}}}{\sqrt{n_{k+1}}e^w}\left(\frac{4}{e} - \frac{2k^2 A_{\tau_{k,w}}}{e^2\sqrt{n_{k+1}}e^w}\right) \\
&\le \frac{\psi(n_{k+1})^2 k^2 A_{\tau_{k,w}}}{\sqrt{n_{k+1}}e^w}\left(\frac{4}{e} - \frac{2}{e^2}\sqrt{e^4 - \frac{(1+\delta)k^4 n_k}{n_{k+1}e^{2w}}}\right) \\
&\le -\psi(n_{k+1})^2 \frac{k^2}{\sqrt{n_{k+1}}e^w} \times \frac{\sqrt{n_{k+1}}e^{w+2}}{k^2} \times \frac{1}{2} \\
&= -\frac{e^2\psi(n_{k+1})^2}{2}.
\end{aligned}
$$

The last inequality holds because $\lim_{k\to\infty} k^4 n_k/n_{k+1} = 0$ and $4/e - 2 < -1/2$. Hence $\mathcal{L}^{\gamma_k e^{-w},k}_{\tau_{k,w}} \to 0$ uniformly in $1 \le w \le \lceil \ln k \rceil$. This implies $\limsup_{k\to\infty} \mathcal{M}^{\gamma_k,k}_{\tau_{k+1}-\tau_k} \le 0$. □

Proposition 4.11. *Let $\omega \in \Omega_C$. Suppose that $\nu_k \le \min(\tau_{k+1}, \sigma_{k,C})$ and*

$$
-A_n \psi^U(A_n^2) \le S_n, \ \ \forall n \in [\tau_k, \nu_k].
$$

Then for sufficiently large k

$$
\mathcal{N}^{\gamma_k,D}_{\nu_k} \ge \frac{\alpha}{2}.
$$

Proof. As in the proof of the previous lemma, we denote $t = n - \tau_k$, $S_t = S_n - S_{\tau_k}$ and $A_t^2 = A_n^2 - A_{\tau_k}^2$. We distinguish two cases:

$$
\text{(a) } A^2_{\nu_k} \le \frac{\psi(n_{k+1})^2}{c_1\gamma^2}, \quad \text{(b) } \frac{\psi(n_{k+1})^2}{c_1\gamma^2} < A^2_{\nu_k} \le A^2_{\tau_{k+1}} - A^2_{\tau_k}.
$$

For case (a), for sufficiently large k and for any $\gamma \le \gamma_k$, as in (58),

$$
\begin{aligned}
\gamma S_{\nu_k} &\le \gamma\left(S_{\nu_k - 1} + c_{\nu_k}\right) \le \gamma\left(\left((1+\delta)A_{\tau_k} + \sqrt{A^2_{\tau_k} + A^2_{\nu_k-1}}\right)\psi(n_{k+1}) + (1+\delta)C\frac{\sqrt{A^2_{\tau_k} + A^2_{\nu_k-1}}}{\psi(A^2_{\tau_k})^3}\right) \\
&\le \psi(n_{k+1})^2\left(\frac{1}{\sqrt{c_1}} + \delta\right)
\end{aligned}
$$

and

$$
\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} 2e^{(1+\delta)^5 e^6 C} e^{\gamma S_{\nu_k}} \to 0 \quad (k \to \infty).
$$

Hence $\mathcal{N}^{\gamma_k,D}_{\nu_k} \ge \alpha/2$ uniformly in $\gamma \in [\gamma_k/k, \gamma_k]$.

For case (b), S_{ν_k} can be evaluated as

$$
\begin{aligned}
S_{\nu_k} &\le S_{\nu_k-1} + c_{\nu_k} \le S_{\nu_k-1} + (1+\delta)C\frac{\sqrt{A^2_{\tau_k} + A^2_{\nu_k-1}}}{\psi(A^2_{\tau_k})^3} \\
&\le \left((1+\delta)A_{\tau_k} + \sqrt{A^2_{\tau_k} + A^2_{\nu_k}}\right)\psi(n_{k+1}) + (1+\delta)C\frac{\sqrt{A^2_{\tau_k} + A^2_{\nu_k}}}{\psi(A^2_{\tau_k})^3}
\end{aligned}
$$

$$\leq \left((1+\delta)A_{\tau_k} + \sqrt{A_{\tau_k}^2 + A_{\nu_k}^2}\left(1 + \frac{(1+\delta)C}{\psi(A_{\tau_k}^2)^3\psi(n_{k+1})}\right)\right)\psi(n_{k+1})$$

by (51). Put

$$q_k^2 := \frac{A_{\tau_k}^2}{A_{\nu_k}^2} \leq \frac{c_1\gamma_k^2}{\psi(n_{k+1})^2}, \qquad s_k := \frac{(1+\delta)C}{\psi(A_{\tau_k}^2)^3\psi(n_{k+1})},$$

so that $\lim_k q_k\psi(n_{k+1})^2 = 0$ and $\lim_k s_k\psi(n_{k+1})^2 = 0$. Then for sufficiently large k

$$\begin{aligned}\frac{S_{\nu_k}^2}{2A_{\nu_k}^2} &\leq \left((1+\delta)^2\frac{q_k^2}{2} + (1+\delta)(1+s_k)q_k\sqrt{1+q_k^2} + (1+s_k)^2\left(\frac{1}{2} + \frac{q_k^2}{2}\right)\right)\psi(n_{k+1})^2 \\ &\leq \frac{\psi(n_{k+1})^2}{2} + \delta.\end{aligned}$$

Then

$$\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \times 2e^{(1+\delta)^5e^6C}e^{S_{\nu_k}^2/(2A_{\nu_k}^2)}\frac{\sqrt{2\pi}}{\gamma A_{\nu_k}} \leq 2e^{(1+\delta)^5e^6C+\delta}\sqrt{2\pi c_1}e^{\delta} \leq \frac{D\alpha}{4}.$$

□

4.6 Dynamic strategy forcing the sharpness

Finally, we prove Proposition 4.1. We assume that by the validity result, Skeptic already employs a strategy forcing $S_n \geq -A_n\psi^U(A_n^2)$ *a.a.* for $\omega \in \Omega_C$. In addition to this strategy, based on Proposition 4.10, consider the following strategy.

> Start with initial capital $\mathcal{K}_0 = \alpha$.
> Set $k = 1$.
> Do the followings repeatedly:
> 1. Apply the strategy in Proposition 4.10 for $n \in [\tau_k, \tau_{k+1}]$.
>   If $\tau_{k+1} < \min(\sigma_{k,C}, \nu_k)$, then go to 2. Otherwise go to 3.
> 2. Let $k = k + 1$. Go to 1.
> 3. Wait until $\exists k'$ such that $-\sqrt{\tau_{k'}}\psi^U(\tau_{k'}) \leq S_{\tau_{k'}} \leq \sqrt{\tau_{k'}}\psi(\tau_{k'})$. Set $k = k'$ and go to 1.

By this strategy Skeptic keeps his capital non-negative for every path ω. For $\omega \in \Omega_0$, $\tau_k = \infty$ for some k and Skeptic stays in Step 1 forever. For $\omega \in \Omega_{=\infty}$, Step 3 is performed infinite number of times, but the overshoot of $|x_n|$ in Step 3 does not make Skeptic bankrupt by Proposition 4.11. Now consider $\omega \in \Omega_C$. Since Skeptic already employs a strategy forcing $S_n \geq -A_n\psi^U(A_n^2)$ *a.a.*, the lower bound in (54) violated only finite number of times. By $\omega \in \Omega_C$, $n \geq \sigma_{k,C}$ is happens only finite number of times. Hence if $S_n \leq A_n\psi(A_n^2)$ *a.a.*, then Step 3 is performed only finite number of times and there exists k_0 such that only Step 2 is repeated for all $k \geq k_0$. Now for each iteration of Step 2, Skeptic multiplies his capital at least by

$$1 + \frac{1-\delta}{D}\lceil \ln k\rceil\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2}.$$

Then

$$\frac{1-\delta}{D}\sum_{k=k_0}^{\infty}\lceil \ln k\rceil\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2} \leq \prod_{k=k_0}^{\infty}\left(1 + \frac{1-\delta}{D}\lceil \ln k\rceil\psi(n_{k+1})e^{-\psi(n_{k+1})^2/2}\right).$$

Since the left-hand side diverges to infinity, the above strategy forces the sharpness.

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